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# A realization of the Kac–Moody algebra on holomorphic fields

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**Abstract.** We discuss the loop group $\mathscr{G}(S^1, G)$ of mappings of the circle $S^1$ into a compact Lie group $G$. We show that there exists a realization of projective unitary representation of $\mathscr{G}$ in the Hilbert space of functionals of holomorphic fields with values in $G^{\mathrm{C}}$ (the complexification of $G$). We show that this representation belongs to the discrete series of positive energy representations of $\mathscr{G}$. We discuss the quantum field theory of holomorphic fields.

## 1. Introduction

Irreducible representations of the loop group and its central extensions have been classified by Kac [1] (see also the review article by Frenkel [2]). A discrete series of unitary positive energy representations (see [3]) has been related to a field-theoretic model of Wess, Zumino and Witten (WZW) [4], which is a realization of the Sugawara current algebra [5] studied in detail by Goddard and Olive [6]. Some representations of the loop algebra have been studied earlier by Gelfand and his collaborators [7, 8] (see also references to their earlier papers) and Albeverio and Hoegh-Krohn [9]. In [7] a representation of the loop group has been realized in the bosonic Fock space. In [8], Gelfand *et al* suggest a representation of the central extension of the loop group in the Fock space assuming that there exists a 2-cocycle. Mickelsson [10] discussed 2-cocycles resulting from the WZW model and the corresponding representations of the Kac–Moody algebra. A realization of the unitary representations of the Kac–Moody algebra in the bosonic Fock space of holomorphic scalar free fields has been discussed by Wakimoto [11] as well as by physicists (see [12–14] and references quoted therein).

In this paper we would like to discuss an approach to the loop group $\mathscr{G}(S^1, G)$ which resembles the construction of the regular representation of a finite-dimensional Lie group $G$. In such a case the representation is constructed by means of the right translation $f(g)\rightarrow\rho(g, h)f(gh)$ (here $\rho$ is a certain multiplier), $f\in L^2(\mathrm{d}\nu)$, where $\nu$ is a quasi-invariant measure on $G$. Such a construction would be interesting in an infinite number of dimensions from the point of view of quantum field theory. Examples of a unitary and covariant transformation of quantum fields under an infinite-dimensional group are rather sparse. However, for representation theory we need to work with fields holomorphically continued from $S^1$ to the disc $\mathbb{D}$. We discuss a measure $\nu$ defined on holomorphic fields. We define a unitary representation of $\mathscr{G}(S^1, G)$ in $L^2(\mathrm{d}\nu)$, which belongs to the discrete series of positive energy representations. This

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represenation could be considered as a particular Hilbert space realization of the representations discussed by Gelfand *et al* [8] and Mickelsson [10]. We discuss in some detail the field-theoretic model of fields with values in $G^{\mathbb{C}}$ (the complexification of $G$). The case of a complex solvable group is explicitly soluble. We discuss the conformal invariance of the model.

## 2. Holomorphic free fields

We consider first the free field defined on the circle $S^1$,

$$\phi(\theta)=\sum_{n=1}^{\infty} a_n \exp(\mathrm{i}n\theta) \tag{2.1}$$

It can be continued analytically to the disk $\mathbb{D}=\{z\in\mathbb{C}: |z|\leqslant 1\}$,

$$\phi(z)=\sum_{n=1}^{\infty} a_n z^n. \tag{2.2}$$

The quantum free field 'at time zero' can be considered as a set of independent complex Gaussian random variables $\{a_n\}$ with the covariance

$$\int \mathrm{d}\nu_0 \bar{a}_n a_m = \delta_{nm}\frac{(2k+m-1)!}{m!(2k-1)!} \equiv \delta_{nm} c_k(n)^{-1} \tag{2.3}$$

($k>0$) defining the Gaussian measure $\nu_0$. We can write $\nu_0$ as a product measure,

$$\mathrm{d}\nu_0(\phi)=\prod_n \mathrm{d}a_n \exp(-c_k(n)|a_n|^2). \tag{2.4}$$

A formal expression for $\nu_0$ can be written in the form

$$\prod_p \mathrm{d}\phi(p)\exp\left(-\int \mathrm{d}\mu_k(p)|\phi(p)|^2\right) \tag{2.5}$$

where $p$ could denote either a point of $S^1$ or of $\mathbb{D}$ and the measure $\mu_k$ is determined by $c_k$ in equation (2.4). The two-point correlation function resulting from the covariance (2.3) is

$$\int \mathrm{d}\nu_0(\phi)\,\overline{\partial\phi(z)}\,\partial\phi(z')=(1-\bar{z}z')^{-2k}. \tag{2.6}$$

By means of the Cayley transformation we can map the disk $\mathbb{D}$ into the upper half-plane $\mathbb{H}$ (see [15]). In such a case the boundary $S^1$ is mapped onto the real line $R$. After this transformation the field $\phi$ for $k=1$ is the positive-energy part of the time-zero massless canonical free field.

## 3. Reproducing kernels and group representations

In the representation theory of groups it is sometimes useful to discuss positive definite kernels instead of the Hilbert space of the representation (see [16]). Let us recall some definitions [17].

Let $Q$ be a set. We say that a function $K$ on $Q\times Q$ is positive definite if for arbitrary $\lambda\in\mathbb{C}$

$$\sum \lambda_i\bar{\lambda}_j K(x_i,x_j)\geq 0. \tag{3.1}$$

There exists a standard method in algebra to construct a linear space $\mathscr{L}$ from any set $Q$. $\mathscr{L}$ consists of 'formal linear combinations of elements of $Q$' (in [17] a less abstract construction is given). So we can define a map

$$v\colon Q\to\mathscr{L}. \tag{3.2}$$

Let us define a subset

$$\mathcal{N}=\{x\in Q\colon K(x,x)=0\} \tag{3.3}$$

and the corresponding linear subspace $\mathcal{N}_{\mathscr{L}}\subset\mathscr{L}$. Then, the positive definite kernel (equation (3.1)) supplies $\mathscr{L}/\mathcal{N}_{\mathscr{L}}$ with the scalar product

$$(v(x),v(y))=K(x,y). \tag{3.4}$$

Now, $\mathscr{L}/\mathcal{N}_{\mathscr{L}}$ equipped with the scaar product (3.4) is the Hilbert space $\mathscr{H}$.

Next, let $Q$ be a measure space, i.e. a $\sigma$-algebra $\Omega$ of measureable sets is chosen and there exists a (non-negative) measure $\nu$ on $\Omega$. We say that $K$ is a reproducing kernel if

$$\int \mathrm{d}\nu(x)K(y,x)K(x,z)=K(y,z). \tag{3.5}$$

Consider an example. Let $\mathscr{F}(M,\mathbb{C}^n)$ be the space of functions defined on $M$ with values in $\mathbb{C}^n$ and $\mu$ a non-negative measure on $M$. Then, the map (3.2) is just an identity. We define a positive definite kernel on $\mathscr{F}(M,\mathbb{C}^n)$ by

$$\mathscr{K}_0^{\mathscr{F}}(u,f)=\exp\left[\sum_{r=1}^{n}\int \mathrm{d}\mu(p)\,\overline{\partial u^r(p)}\,\partial f^r(p)\right]. \tag{3.6}$$

$\mathscr{K}_0^{\mathscr{F}}$ is a reproducing kernel. Equation (3.6) is in fact a rigorous definition of the Gaussian measure $\nu_0$ (equation (2.4)) (when $M=\mathbb{D}$ and $\mu=\mu_k$).

We can construct new positive definite kernels from a known one. Let $K$ be a positive definite kernel on $M$ and

$$\sigma\colon Q\to M$$

an invertible map. Denoting $x_1=\sigma^{-1}(m_1)$ and $x_2=\sigma^{-1}(m_2)$ we define a kernel $K_\sigma$ on $Q$,

$$K_\sigma(x_1,x_2)\equiv K(\sigma(x_1),\sigma(x_2)). \tag{3.7}$$

It is an easy check that $K_\sigma$ is positive definite:

$$\sum \lambda_k\bar{\lambda}_r K_\sigma(x_k,x_r)=\sum \lambda_k\bar{\lambda}_r K(\sigma(x_k),\sigma(x_r))=\sum \lambda_k\bar{\lambda}_r K(m_k,m_r)\geq 0.$$

If $K$ is the reproducing kernel with the reproducing measure $\nu$ in equation (3.5) then $K_\sigma$ is the reproducing kernel with the reproducing measure $\nu_\sigma=\nu\circ\sigma$, where we define

$$\nu_\sigma(A)=\nu(\sigma(A)) \tag{3.8}$$

for any set $A$ being a $\sigma^{-1}$-image of a measureable set. In fact, we have

$$\int \mathrm{d}\nu_\sigma(x)K_\sigma(x_1,x)K_\sigma(x,x_2)$$

$$=\int \mathrm{d}\nu(\sigma(x))K(\sigma(x_1),\sigma(x))K(\sigma(x),\sigma(x_2))$$

$$=\int \mathrm{d}\nu(m)K(\sigma(x_1),m)K(m,\sigma(x_2))=K(\sigma(x_1),\sigma(x_2))=K_\sigma(x_1,x_2).$$

We assume that a transformation group $G$ is defined on $Q$ (we denote its action by $xg$). We say that the kernel $K$ is projectively invariant if for each $g\in G$

$$K(xg,yg)=\overline{\Lambda(g,x)}\Lambda(g,y)K(x,y). \tag{3.9}$$

If the kernel is projectively invariant then there exists in $\mathcal{H}$ a unitary (in general projective) representation $U(G)$ of the group $G$ defined by

$$U_g v(x)=\Lambda(g,x)^{-1}v(xg). \tag{3.10}$$

Note that $U_{g_1}=U_{g_2}$ if $v(g_1)-v(g_2)\in\mathcal{N}_{\mathcal{L}}$ and $\Lambda(g_1,x)=\Lambda(g_2,x)$.

It can be shown (and calculated from $\Lambda$) that there exists a two-cocycle $\gamma(|\gamma|=1)$ such that

$$U(g_1)U(g_2)=\gamma(g_1,g_2)\,U(g_1g_2). \tag{3.11}$$

The cocycle condition resulting from associativity reads

$$\gamma(g_1,g_2)\gamma(g_1g_2,g_3)=\gamma(g_1,g_2g_3)\gamma(g_2,g_3). \tag{3.12}$$

## 4. Fields with values in a group

We now apply the method of construction of reproducing kernels and reproducing measures expressed by equations (3.7) and (3.8). We begin with the reproducing kernel $\mathcal{K}_0^{\mathcal{F}}$ (equation (3.6) with $M=\mathbb{D}$). Let us consider a complex vector bundle $V(\mathbb{D},V_n)$ over $\mathbb{D}$. Let $\mathcal{H}(\mathbb{D},V_n)$ be the vector space of local holomorphic sections of $V(\mathbb{D},V_n)$. If $\phi_i\in\mathcal{H}(\mathbb{D},V_n)$, then we define

$$\mathcal{K}_0^{\mathcal{F}}(\phi_1,\phi_2)=\exp\left[\frac{1}{\pi}\sum_{a=1}^{n}\int_{\mathbb{D}}\mathrm{d}^2z\,\overline{\partial\phi_1^a(z)}\,\partial\phi_2^a(z)\right]. \tag{4.1}$$

The reproducing measure is defined rigorously by the reproducing property (3.5) and formally by

$$\mathrm{d}\nu_0(\phi)\sim\prod_z \mathrm{d}\phi^a(z)\exp\left[-\frac{1}{\pi}\int_{\mathbb{D}}\mathrm{d}^2z\,\overline{\partial\phi^a(z)}\,\partial\phi^a(z)\right]. \tag{4.2}$$

This is the Gaussian measure of independent free fields $\phi^a$ with the two-point function defined in equation (2.6) with $k=1$.

According to equation (3.7) if $f$ is a (locally) invertible map of a set $\mathcal{T}$ (at this stage we have some freedom in the choice of $f$ and $\mathcal{T}$)

$$f\colon \mathcal{T}\to\mathcal{H}(\mathbb{D},V_n) \tag{4.3}$$

expressed in coordinates as

$$f^a(\psi)(w)=\kappa^{-1/2}\,\partial\phi^a(w) \tag{4.4}$$

then

$$\mathcal{K}_f(\psi_1,\psi_2)=\exp\left[\frac{\kappa}{\pi}\sum_{a=1}^{n}\int_{\mathbb{D}} d^2z\,\overline{f^a(\psi_1)(z)}\,f^a(\psi_2)(z)\right] \tag{4.5}$$

is the positive definite reproducing kernel on $\mathcal{T}$.

Let $V_n=T_1G=\mathcal{A}$ now be the Lie algebra of a compact semi-simple Lie group $G$. Let $G^{\mathbb{C}}$ be its complexification and $\mathcal{A}^{\mathbb{C}}$ the complexification of $\mathcal{A}$. Let $\mathcal{G}(\mathbb{D},G^{\mathbb{C}})$ (the set of maps from $\mathbb{D}$ into $G^{\mathbb{C}}$) be the set $\mathcal{T}$ in equation (4.3), we consider the map

$$f\colon\mathcal{G}(\mathbb{D},G^{\mathbb{C}})\to\mathcal{H}(\mathbb{D},\mathcal{A}^{\mathbb{C}}) \tag{4.6}$$

defined by

$$f(g)=g^{-1}\,\partial g. \tag{4.7}$$

The inverse map $f^{-1}$ is defined as the solution of the equation (an ordinary differential equation in the complex domain $\mathbb{D}$)

$$g^{-1}\,\partial g=\frac{1}{\sqrt{\kappa}}\,\partial\phi \tag{4.8}$$

mapping 0 of $\mathcal{A}^{\mathbb{C}}$ into the unit element of $G^{\mathbb{C}}$ (here $\phi\in\mathcal{H}(\mathbb{D},\mathcal{A}^{\mathbb{C}})$ are $n$ independent holomorphic free fields (2.6) corresponding to $k=1$).

From equation (4.5) we get a positive definite kernel on $\mathcal{G}(\mathbb{D},G^{\mathbb{C}})$,

$$\mathcal{K}^{\mathcal{G}}(g_1,g_2)=\exp\left[\frac{\kappa}{\pi}\int_{\mathbb{D}} d^2z\langle(g_1^{-1}\,\partial g_1)^+,g_2^{-1}\,\partial g_2\rangle\right] \tag{4.9}$$

where $\langle,\rangle$ is the Killing form on $\mathcal{A}$ (positive definite).

The kernel $\mathcal{K}^{\mathcal{G}}$ is projectively invariant under the group $\mathcal{G}(\mathbb{D},G)$ of real analytic maps $\mathbb{D}\to G$. For the multiplier $\Lambda$ (equation (3.9)) we get

$$\Lambda(h,g)=\exp\left[\frac{\kappa}{\pi}\int_{\mathbb{D}} d^2z(\tfrac{1}{2}\langle(h^{-1}\,\partial h)^+,h^{-1}\,\partial h\rangle+\langle(\partial hh^{-1})^+,g^{-1}\,\partial g\rangle)\right] \tag{4.10}$$

where $h\in\mathcal{G}(\mathbb{D},G)$ and $g\in\mathcal{G}(\mathbb{D},G^{\mathbb{C}})$.

Clearly, $h\in\mathcal{G}(\mathbb{D},G)$ does not map $\mathcal{G}(\mathbb{D},G^{\mathbb{C}})$ onto itself. The formalism of section 3 still applies. We need to extend the definition of the kernel $\mathcal{K}^{\mathcal{G}}$ (see equation (4.9)) to the orbit of $\mathcal{G}(\mathbb{D},G^{\mathbb{C}})$ under the action of $\mathcal{G}(\mathbb{D},G)$ denoted by $\mathcal{G}_{\mathbb{C}}(\mathbb{D},G)$. The kernel (4.9) remains positively definite on $\mathcal{G}_{\mathbb{C}}(\mathbb{D},G)$. Then, equation (3.10) defines a unitary projective representation of the group $\mathcal{G}(\mathbb{D},G)$. A simple calculation gives the formula for the 2-cocycle $\gamma$ (equation (3.11)),

$$\gamma(h_1,h_2)=\exp\left[\frac{\kappa}{2\pi}\int_{\mathbb{D}} d^2z(\langle(h_1^{-1}\,\partial h_1)^+,\partial h_2h_2^{-1}\rangle-\langle(\partial h_2h_2^{-1})^+,h_1^{-1}\,\partial h_1\rangle)\right]. \tag{4.11}$$

We shall show that formula (3.10) defined by the kernel (4.9) defines a representation of the central extension of the loop algebra. In order to see this let us consider infinitesimal transformations $h=\exp(\varepsilon\mathfrak{h})$,

$$\partial hh^{-1}=\varepsilon\partial\mathfrak{h}. \tag{4.12}$$

Hence, neglecting higher orders in $\varepsilon$,

$$\Lambda(h, g) = \exp\left[\frac{\kappa\varepsilon}{\pi}\int_{\mathbb{D}} \mathrm{d}^2 z\, \bar{\partial}(\mathfrak{h}^+ g^{-1}\,\partial g)\right]$$

because $g$ is a holomorphic function, i.e. $\Lambda(h, g)$ depends only on the values of $h$ and $g$ on the boundary $\partial\mathbb{D} = S^1$. If $\mathfrak{h} = 0$ on $\partial\mathbb{D}$ then $\Lambda(h, g) = 1$ (at least for infinitesimal transformations $h$). Moreover, if $\mathfrak{h} = 0$ on $\partial\mathbb{D}$ then $v(gh) - v(g) \in \mathcal{N}_{\mathcal{L}}$. In order to prove this let us note that

$$(gh)^{-1}\,\partial gh \simeq g^{-1}\,\partial g + \varepsilon[g^{-1}\,\partial g, \mathfrak{h}] + \varepsilon\,\partial\mathfrak{h}. \tag{4.13}$$

Inserting equation (4.13) into equation (4.9) we can see that because of holomorphicity of $g$ only the value of $\mathfrak{h}$ on the boundary of $\mathbb{D}$ contributes to $\mathcal{K}^{\mathcal{G}}$. Hence, if $\mathfrak{h} = 0$ on $\partial\mathbb{D}$ then $v(gh) - v(g) \in \mathcal{N}_{\mathcal{L}}$ (see the definition below equation (3.3)). As a consequence, if $h = 1$ on $\partial\mathbb{D}$ then from equation (3.10) $U_h = 1$ (at least for infinitesimal transformations). Now, the loop group $\mathcal{G}(S^1, G)$ has a realization (see [10]) as a quotient group $\mathcal{G}(\mathbb{D}, G)/\mathcal{G}_1(\mathbb{D}, G)$, where $\mathcal{G}_1$ denotes the set of maps, which are 1 on the boundary $\partial\mathbb{D}$. We have just shown that the subgroup $\mathcal{G}_1(\mathbb{D}, G)$ has a trivial representation of its algebra in the Hilbert space determined by the kernel (4.9). Hence, formula (3.10) defines a representation of the central extension of the loop algebra determined by the 2-cocycle $\gamma$ (equation (4.11)). The 2-cocycle $\gamma$ is related to the WZW action. When we extend the field and the integration domain to the whole compactified plane $\mathbb{C}$, then this extended 2-cocycle $\gamma_{\mathbb{C}}$ can be expressed as

$$\gamma_{\mathbb{C}}(h_1, h_2) = \exp[W(h_1 h_2) - W(h_1) - W(h_2) - \overline{W(h_1 h_2)} + \overline{W(h_1)} + \overline{W(h_2)}] \tag{4.14}$$

where $W$ is the WZW action [4, 19]. The equality of equations (4.11) and (4.14) follows from the Polyakov–Wiegmann formula [19] fpr $W(h_1 h_2)$. The imaginary part of $W(h)$ in equation (4.14) is equal to

$$\omega = \frac{\kappa}{\pi}\int_B (h^{-1}\,\mathrm{d}h)^3$$

where $h$ is a smooth extension of $h$ to a ball $B$ with $\mathbb{C}$ as its boundary. It is known that $\omega$ is an integer if $\kappa = k/12^{\delta^2}$, where $k$ is an integer and $\delta$ is the length of the maximal root of $\mathcal{A}$. Only a discrete set of values for $\kappa$ is allowed. This follows from the requirement of $\gamma$ to be single valued. The 2-cocycle $\gamma$ (4.11) as a central extension of the loop group has been discussed in [10] and [18].

Consider now the reproducing measure $\nu$. According to formula (3.8) this measure is equal to $\nu = \nu_f = \nu_0 \circ f$, where the map $f$ is defined in equation (4.7) and $\nu_0$ is the Gaussian measure (2.6). Formula (3.8) defining $\nu_f$ as a transformation of $\nu_0$ is a mathematically precise definition of the reproducing measure $\nu$. However, in order to relate it to the Lagrangian formalism we derive its formal expression. We have from equations (4.2) and (4.7)

$$\begin{aligned}\mathrm{d}\nu_f(\psi) &= \mathrm{d}\nu_0(f(\psi)) = \mathrm{d}f^a(\psi)\exp\left[-\frac{1}{\pi}\int \overline{f^a(\psi)} f^a(\psi)\right] \\ &= \mathrm{d}\psi^a \det\left|\frac{\partial f^c}{\partial \psi^a}\right| \exp\left[-\frac{\kappa}{\pi}\int \mathrm{d}^2 z \langle (g^{-1}\,\partial g)^+, g^{-1}\,\partial g\rangle\right].\end{aligned} \tag{4.15}$$

The Maurer–Cartan form can be expanded into a basis $\{\tau_a\}$ of $\mathscr{A}$

$$g^{-1}\partial g=M^a_\alpha\tau_a\,\partial\psi^\alpha. \tag{4.16}$$

From equation (4.8) we get for the Jacobian in equation (4.15)

$$J^a_\beta=\frac{\delta f^a(z)}{\delta\psi^\beta(z')}=M^a_\beta(\psi)\,\partial\delta(z-z')+\partial_\beta M^a_\alpha(\psi)\,\partial\psi^\alpha(z)\delta(z-z'). \tag{4.17}$$

Let

$$L^\alpha_a M^a_\beta=\delta^\alpha_\beta$$

be the inverse of $M$. We transform the operator $J$ (equation (4.17)) into an operator $\mathscr{J}$: $TG\rightarrow TG$,

$$\mathscr{J}^\sigma_\alpha=J^a_\alpha L^\sigma_a=\delta^\sigma_\alpha\partial-T^\sigma_{\alpha\beta}(\psi)\,\partial\psi^\beta \tag{4.18}$$

where

$$T^\sigma_{\alpha\beta}=M^a_\beta\,\partial_\alpha L^\sigma_a-M^a_\alpha\,\partial_\beta L^\sigma_a=f^c_{ab}L^\sigma_c M^a_\alpha M^b_\beta \tag{4.19}$$

where in the last step in equation (4.19) the fact that $\{L^a\}$ form the basis of the Lie algebra (with structure constants $f^c_{ab}$) of left invariant vector fields has been used. The transformation $J\rightarrow\mathscr{J}$ has the Jacobian $(\det L)^{-1}$. We have

$$\mathrm{d}\psi(\det L)^{-1}=\mathrm{d}g(\psi)$$

where d$g$ is the Haar measure on $G$. Let $\eta_{ab}=\langle\tau_a,\tau_b\rangle$ be the Killing tensor. Define

$$h_{\alpha\bar\sigma}=M^a_\alpha\overline{M^b_\sigma}\eta_{ab}. \tag{4.20}$$

Then, the measure $\nu_f$ (equation (4.15)) can be expressed in the form

$$\begin{aligned}\mathrm{d}\nu_f(\psi)&=\mathrm{d}g(\psi)\det[\mathscr{J}]\exp\left[-\frac{\kappa}{\pi}\int\mathrm{d}^2z\,h_{\alpha\bar\sigma}(\psi)\,\partial\psi^\alpha\,\overline{\partial\psi^\sigma}\right]\\&=\mathrm{d}g(\psi)\,\mathrm{d}\chi\exp\left[-\int\mathscr{L}(\psi,\chi)\right]\end{aligned} \tag{4.21}$$

where $\chi$ is a Dirac (anticommuting) spinor field and

$$\mathscr{L}(\psi,\chi)=h_{\alpha\bar\sigma}(\psi)\,\partial\psi^\alpha\,\overline{\partial\psi^\sigma}+\bar\chi_\alpha\tfrac{1}{2}(1-\gamma_5)\gamma^\mu(\delta^\alpha_\beta\,\partial_\mu-T^\alpha_{\beta_\rho}\,\partial_\mu\psi^\rho)\chi^\beta. \tag{4.22}$$

The determinant in equation (4.21) can be computed [20, 21]. It is equal to its holomorphic anomaly,

$$\det[\mathscr{J}]=\exp\left[-\frac{1}{4\pi}\int T^\alpha_{\beta\sigma}(\psi)\overline{T^\beta_{\alpha\rho}}(\psi)\,\partial\psi^\sigma\,\overline{\partial\psi^\rho}\right]. \tag{4.23}$$

The effective action resulting from equations (4.21) and (4.23) leads to non-compact WZW Lagrangians discussed in [22–25], but now with holomorphic $\sigma$-fields. The bosonic part of the Lagrangian (4.22) is equal to the area of the surface in $G^{\mathbb{C}}$ resulting as the $f$-image (4.7) of $\mathbb{D}$. Models of this type could have applications in string theory.

The group-theoretical content of the holomorphic field theory can be reduced to that of the reproducing kernel (4.9). The Hilbert space $\mathscr{H}$ defined by equation (3.4) has a realization as $L^2(\mathrm{d}\nu_f)$. Let us call the measure $\nu_f$ quasi-invariant under right group translations $h \in \mathscr{G}(\mathbb{D}, G)$ if for any $F \in L^2(\mathrm{d}\nu_f)$

$$\int d\nu_f(g) F(gh^{-1}) = \int \mathrm{d}\nu_f(g) |\Lambda(g, h)|^{-2} F(g). \tag{4.24}$$

Then, we can prove the formula (4.24) using the reproducing property (3.5) in the form

$$\int \mathrm{d}\nu_f(gh) \mathcal{K}^{\mathscr{G}}(g_1 h, gh) \mathcal{K}^{\mathscr{G}}(gh, g_2 h) = \mathcal{K}^{\mathscr{G}}(g_1 h, g_2 h). \tag{4.25}$$

Applying the transformation property (3.9) we see that equation (4.24) holds true when applied to functionals of the form of the exponentials (4.9) (we expect that such functionals form a dense set in $L^2(\mathrm{d}\nu_f)$).

From equations (4.24) and (3.10) it follows that the representation of $\mathscr{G}(\mathbb{D}, G)$ defined in $L^2(\mathrm{d}\nu_f)$ by

$$(U_h F)(g) = \Lambda(g, h)^{-1} F(gh) \tag{4.26}$$

(where $h \in \mathscr{G}(\mathbb{D}, G)$) on functionals $F$ of holomorphic $G^{\mathbb{C}}$-valued fields ($g \in \mathscr{G}(\mathbb{D}, G^{\mathbb{C}})$) is unitary and equivalent to the representation (3.10) derived from the positive definite kernel $\mathcal{K}^{\mathscr{G}}$ (equation (4.9)).

The pointwise multiplication $gh$ in equation (4.26) when expressed in terms of $\phi$ (equation (4.8)) takes the form

$$\partial\phi \rightarrow h^{-1}\,\partial\phi h + h^{-1}\,\partial h. \tag{4.27}$$

Formula (4.27) relates the representation (4.26) to the central extension of the loop algebra suggested at the end of the paper of Gelfand *et al* [8] (although these authors do not consider holomorphic fields at all).

## 5. A soluble model

Equations (4.8) can easily be solved when $G$ is a solvable group. We are interested in this section in conformal invariance of the solution. For a discussion of the conformal invariance it is convenient to map the disk $\mathbb{D}$ onto the upper half-plane $\mathbb{H}$ by means of the Cayley transformation

$$z = (1 + \mathrm{i}w)(1 - \mathrm{i}w)^{-1}. \tag{5.1}$$

The correlation function (2.6) transforms into

$$\int \mathrm{d}\nu_0(\phi)\, \overline{\partial\phi(w)}\, \partial\phi(w') = \frac{1}{4}\left[-\frac{\mathrm{i}}{2}(w' - \bar{w})\right]^{-2k} \tag{5.2}$$

where $w, w' \in \mathbb{H}$. We consider here the simplest case of equation (4.8), a solvable $2 \times 2$ unimodular matrix. Then,

$$g = NA$$

where

$$N = \begin{bmatrix} 1 & 0 \\ Z & 1 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} \mathrm{e}^{\psi} & 0 \\ 0 & \mathrm{e}^{-\psi} \end{bmatrix}.$$

$\phi$ is a matrix,

$$\phi = \begin{bmatrix} \phi_0 & 0 \\ \phi_1 & -\phi_0 \end{bmatrix}$$

whose entries are independent free fields with the two-point function defined in equation (5.2) with $k=1$. Equations (48) read

$$\begin{aligned} \partial\psi &= \sqrt{\sigma}\,\partial\phi_0 \\ \partial Z &= \sqrt{\sigma}\exp(-2\psi)\,\partial\phi_1 \end{aligned} \tag{5.3}$$

where we denoted $\sigma = 1/\kappa$. The general solution of equations (5.3) is

$$\psi(w) = \sqrt{\sigma}\int_\alpha^w \partial\phi_0(z)\,\mathrm{d}z \qquad Z(w) = \sqrt{\sigma}\int_\beta^w \exp[-2\psi(\zeta)]\,\partial\phi_1(\zeta)\,\mathrm{d}\zeta. \tag{5.4}$$

It depends on the parameters $\alpha$ and $\beta$.

The measure $\nu_f$ (equation (4.21)) after the calculation of the determinant (4.23) is

$$\mathrm{d}\nu = \mathrm{d}Z\,\mathrm{d}\psi\exp\left[-\frac{\kappa+1}{\pi}\int_{\mathrm{H}} \mathrm{d}^2w\,\partial\psi\,\overline{\partial\psi} - \frac{\kappa}{\pi}\int_{\mathrm{H}} \mathrm{d}^2w\exp(2\psi+\overline{2\psi})\,\partial Z\,\overline{\partial Z}\right]$$

which is a holomorphic version of a measure discussed in [22]. We consider the field

$$\Phi_\alpha(w) = \exp(\psi(w)+\bar{\psi}(w))Z(w) \tag{5.5}$$

as a candidate for a conformal covariant field.

Let us first compute the two-point function of $\psi$. From equation (5.2) ($k=1$) we get

$$\Omega_\alpha(w_1, w_2) \equiv \langle\overline{\psi(w_1)}\psi(w_2)\rangle = \sigma(-\ln(w_2-\overline{w_1}) + \ln(\alpha-\overline{w_1}) + \ln(w_2-\bar{\alpha}) - \ln(\alpha-\bar{\alpha})). \tag{5.6}$$

The two-point function $\Omega_\alpha$ is conformally invariant with a scale dimension equal to zero if $\alpha$ undergoes a conformal transformation together with $w$. In such a case the parameter $\alpha$ in $\Phi_\alpha$ also undergoes a conformal transformation. Another way out of this well known conformal anomaly [26] is to restrict the field $\psi$ to test functions $f$ such that $\int \mathrm{d}^2w f(w,\bar{w}) = 0$. In such a case only the first term on the right-hand side of equation (5.6) is relevant. Let us introduce

$$\Omega(w_1, w_2) \equiv \langle\overline{\Psi(w_1)}\Psi(w_2)\rangle = -\sigma\ln(\mathrm{w}_2 - \bar{w}_1) \tag{5.7}$$

as a definition of $\Psi$. $\Psi$ can be expressed by $\psi$ as $\Psi(w) = \psi(w) + \mathrm{i}\psi(\alpha)$. We express $\psi$ in formula (5.4) by $\Psi$, and dividing by $\exp[\mathrm{i}\psi(\alpha) - \mathrm{i}\bar{\psi}(\alpha)]$ we modify definition (5.5),

$$\Phi(w) = \exp[-\langle\overline{\Psi(w)}\Psi(w)\rangle]\exp[\Psi(w)+\overline{\Psi(w)}]\int_\beta^w \exp[-2\Psi(\zeta)]\,\partial\phi_1(\zeta)\,\mathrm{d}\zeta. \tag{5.8}$$

The two-point function of $\Phi$ is

$$\begin{aligned} \langle\overline{\Phi(w_1)}\Phi(w_2)\rangle = &-\sigma\int_{\bar{\beta}}^{\overline{w_1}}\int_\beta^{w_2} \mathrm{d}\zeta_2\,\mathrm{d}\overline{\zeta_1}(\zeta_2-\overline{\zeta_1})^{-2} \\ &\langle\exp[\overline{\Psi(w_1)} + \Psi(w_1) - 2\overline{\Psi(\zeta_1)} + \overline{\Psi(w_2)} + \Psi(w_2) - 2\Psi(\zeta_2))]\rangle. \end{aligned} \tag{5.9}$$

The last expectation value is equal to

$$\begin{aligned} \langle\ldots\rangle = \exp[&-2\Omega(\zeta_1, w_1) + \Omega(w_2, w_1) + \Omega(w_1, w_2) - 2\Omega(\zeta_1, w_2) \\ &-2\Omega(w_1, \zeta_2) + 4\Omega(\zeta_1, \zeta_2) - 2\Omega(w_2, \zeta_2)]. \end{aligned} \tag{5.10}$$

We can see from equations (5.9) and (5.10) that if we formally let $\beta \to \infty$, then the correlation functions are scale covariant with the scaling dimension $d=-\sigma$ (the negative sign means that the two-point function grows with the distance as $|w_1-w_2|^\sigma$). If $\beta=\infty$, then the integral (5.9) is divergent. We could define the integral (5.9) by an analytic continuation in $\sigma$. If $\sigma$ is negative then the integral (5.9) is convergent even if $\beta=\infty$. A simple Gaussian functional integral gives a formula for $n$-point correlation functions. If $\beta=\infty$ the correlation functions are formally scale covariant with the scaling dimension $d=-\sigma$.

The correlation functions which we get this way resemble the Feigin–Fuchs representation of conformal field theory [27]. They appear to be related to the correlation functions of non-compact wzw models discussed in [22–25] (if we express the plane integrals there by line integrals according to [28]). The negative dimension may be a result of non-compactness of the solvable model.

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